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Entropy of flows, revisited

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Abstract. We introduce a concept of measure-theoretic entropy for flows and study its invariance under measure-theoretic equivalences. Invariance properties of the corresponding topological entropy is studied too. We also answer a question posed by Bowen-Walters in [**3**] concerning the equality between the topological entropy of the time-one map of an expansive flow and the time-one map of its symbolic suspension.

Keywords: Entropy, measure-theoretic entropy, topological entropy, invariant measure.

1. Introduction

In the context of dynamical systems it is understood that a reasonable measure-theoretic or topological entropy should be a measure of the complexity of the system and they should be invariant under measurable or topological change of coordinates, respectively. If the dynamical system is a homeomorphism on a compact manifold, Kolmogorov and Sinai found successfully a good concept for measure-theoretic entropy. Nevertheless if the dynamical system is a flow we face some difficulties, the entropy of the homeomorphisms generated by the time-one map of two measure-theoretically or topologically equivalent flows may not be the same. The main problem is that in general measurable and topological change of coordinates (in the case of flows) allow speed changes which are hard to be taken in account. Here we introduce a concept of measure-theoretic entropy (and topological entropy) for flows which behaves reasonable well when we make a speed change or a reparametrization of the flow. It is easy to study its invariance, in particular the

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0 and ∞ entropy are preserved under measure-theoretic equivalence or speed changes.

Instead of adapting time-one map, our concept of measure-theoretic and topological entropy focus on the whole flow itself. Iterating partition as in the discrete case does not work here and so we consider certain open sets consisting of points whose reparametrized segments of orbits are close each other. The measure-theoretic and topological entropy obtained generalizes the original ones defined usually by time-one map. We prove that they coincide in the special case of flows without fixed points.

In [**10**] it was introduced a concept of topological entropy for flows which takes in consideration all possible reparametrizations of the flow. Here we will follow some ideas in [**5**], [**10**] to introduce our concept of measure-theoretic entropy for flows. The corresponding topological entropy is studied too.

In [**3**] Bowen-Walters posed a question concerning the equality between the topological entropy of the time-one map of an expansive flow and the topological entropy of the time-one map of its symbolic suspension. In the present paper we answer this question positively. Before, it was answered positively by Bowen in [**2**] in the case of Axiom A flows.

2. Basic Concepts and Main Results

We start this section introducing some notation. Let (M, d) denote a compact metric space and $\phi: R \times M \to M$ (or just ϕ if clear) a continuous flow on M. For $t \in R$, $\phi_t: M \to M$ denotes the homeomorphism given by $\phi_t(x) = \phi(x, t)$. A Borel probability measure (probability for short) is called *ϕ_t-invariant* if for any Borel set B it holds $\mu(\phi_t(B)) = \mu(B)$. It is called *ϕ-invariant* if it is ϕ_t-invariant for all t. As usual a ϕ_t-invariant probability is called *ergodic* if any ϕ_t-invariant Borel set has measure 0 or 1. A ϕ-invariant probability is called *ergodic* if any Borel set ϕ_t-invariant for any t has measure 0 or 1. The set of all ergodic ϕ_t-invariant and the set of all ergodic ϕ-invariant probabilities are denoted respectively by $\mathcal{E}_{\phi_t}$ and $\mathcal{E}_{\phi}$.

Given a closed interval I which contains zero, a continuous map $\alpha: I \to R$ is called a *reparametrization* if it is an increasing homeomorphism onto its image and $\alpha(0) = 0$. The set of all such reparametrizations is denoted by $Rep(I)$. Given a flow ϕ on M, $x \in M$, $t \in R$ and $\epsilon > 0$ we set

$$B(x, t, \epsilon, \phi) := \{y \in M; \text{ there exists } \alpha \in Rep[0, t] \text{ with, } d(\phi_{\alpha(s)}x, \phi_s y) < \epsilon, 0 \le s \le t\}$$

and call it a (t, ϵ, ϕ)*-ball.* Clearly, the (t, ϵ, ϕ)-balls are open sets.

Let us introduce now a concept of measure-theoretic entropy for flows.

Definition 1 *Given a flow ϕ on M, $\mu \in \mathcal{E}_\phi$ and $\delta \in (0,1)$. Let $N(\delta, t, \epsilon, \phi)$ denotes the smallest number of (t, ϵ, ϕ)-balls needed to cover a set whose μ-probability is bigger than $1-\delta$. Then the* measure-theoretic entropy of *ϕ, denoted by $e_\mu(\phi)$, is defined by*

$$e_\mu(\phi) := \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log N(\delta, t, \epsilon, \phi).$$

We remark that the limit above is not dependent on the choice of δ, see [**1**], [**5**]. *The* topological entropy of ϕ , *denoted by $e(\phi)$, is defined by*

$$e(\phi) := \sup\{e_\mu(\phi); \mu \in \mathcal{E}_\phi\}.$$

Definition 2 *Let $\phi: R \times M \to M$ and $\psi: R \times W \to W$ be two flows on compact metric spaces with ergodic invariant probabilities μ and ν, respectively. We say that (M, ϕ, μ) is* measure-theoretically equivalent *to (W, ψ, ν) if there exist a measure preserving homeomorphism $P: M \to W$ and a continuous map $\sigma: R \times M \to R$ satisfying the following*

1. *$\sigma_x: R \to R$ is strictly increasing for all $x \in M$;*
2. *$\sigma_x(s+t) = \sigma_x(s) + \sigma_{\phi_s(x)}(t)$, for all $x \in M$ and $s, t \in R$;*
3. *$P \circ \phi_t(x) = \psi_{\sigma_x(t)} \circ P(x)$, for all $x \in M$ and $t \in R$.*

The continuous map σ is called a cocycle of ϕ. If ϕ, ψ are just topological flows we say that ψ is a generalized time change of ϕ if there exist a homeomorphism $P: M \to W$ and a continuous map σ as above.

Measure-theoretic equivalence is an equivalence relation, that is: it is symmetric, reflexive and transitive (see Lemma 2 in the next section). Let us recall that two flows ϕ, ψ are called *topologically equivalent* if there exists a homeomorphism $H\colon M \to W$ which maps orbits of ϕ onto orbits of ψ preserving their orientation. We remark that two flows which do not have fixed points are topologically equivalent iff one is a generalized time change of the other. Nevertheless, there exist topologically equivalent flows which are not a generalized time change one of the other. This fact follows easily from Lemma 1 in the next section. The following theorem states that the measure-theoretic entropy defined above is in some extent invariant under measure-theoretic equivalence.

Theorem 1. *Let (M, ϕ, μ) and (W, ψ, ν) be measure-theoretically equivalent flows where μ, ν are ergodic. Then $e_\mu(\phi) = 0$ iff $e_\nu(\psi) = 0$ and $e_\mu(\phi) = \infty$ iff $e_\nu(\psi) = \infty$.*

Given a flow ϕ we denote, respectively, by $h_\mu(\phi_t)$ and $h(\phi_t)$ the usual measure-theoretic entropy and topological entropy of the homeomorphism ϕ_t. The next theorem relates the entropy we introduce above with these ones.

Theorem 2. *If ϕ is a continuous flow as above which has an ergodic invariant probability μ, then $e_\mu(\phi) \leq h_\mu(\phi_1)$. If ϕ has no fixed points the equality holds.*

The corresponding results hold for topological entropy.

Theorem 3. *Let ϕ, ψ be two flows on compact metric spaces. If these flows are a generalized time change one of the other, then the following hold:*

1. *$e(\phi) = 0$ iff $e(\psi) = 0$ and $e(\phi) = \infty$ iff $e(\psi) = \infty$.*
2. *$e(\phi) \leq h(\phi_1)$ and the equality holds when ϕ has no fixed point.*

In [8] it is proved that Part 1 of Theorem 3 is still true if we replace $e(\phi)$ by $h(\phi_1)$ (and $e(\psi)$ by $h(\psi_1)$). Nevertheless an example of two topologically equivalent flows ϕ, ψ such that $h(\phi_1) = 0$ and $h(\psi_1) > 0$ is given. Note that, in the special case of flows without fixed points, the new measure-theoretic entropy and the new topological entropy we introduced coincide with the measure-theoretic entropy and the topological entropy (respectively) given by the time-one map.

Given an expansive flow ϕ one can define a symbolic suspension flow associated to it which we denote here by φ, see Section 6 and [**3**] for the precise definition. The following theorem answer a question posed in [**3**].

Theorem 4. *Let ϕ be an expansive flow without fixed points and φ be a symbolic suspension flow for ϕ, then $h(\phi_1) = h(\varphi_1)$.*

3. Preliminary Facts

In this section we start establishing some intermediate steps to prove the theorems stated in the previous section.

Lemma 1. *If σ is a cocycle of a flow ϕ there exist constants M_1, M_2 such that $M_1 t \leq \sigma_x(t) \leq M_2 t$, for all $|t| \geq 1$.*

Proof. See [**10**].

Lemma 2. *Measure-theoretic equivalence of flows is a reflexive, symmetric and transitive relation.*

Proof. If (M, ϕ, μ) is measure-theoretically equivalent to (W, ψ, ν), then (W, ψ, ν) is measure-theoretically equivalent to (M, ϕ, μ). Indeed, we define $\lambda: R \times W \to R$ by $\lambda_y = \sigma_x^{-1}$, where $y = P(x)$. Then λ is continuous and satisfies Property 1.-3. This proves that measure-theoretic equivalence of flows is a symmetric relation. That it is reflexive and transitive is immediate, see [**12**]. □

Lemma 3. *Let ϕ be a flow without fixed points on a compact metric space M. Then for any given $\epsilon_1 > 0$ there exists $\epsilon > 0$ such that for any x, $y \in M$ and any reparametrization $\alpha \in Rep(I)$, if $d(\phi_{\alpha(s)}(x), \phi_s(y)) < \epsilon$ for all $s \in I$ it holds $|\alpha(s) - s| < \epsilon_1$ whenever $|s| \leq 1$ and $|\alpha(s) - s| \leq |s|\epsilon_1$ whenever $|s| > 1$.*

Proof. See [**10**].

Lemma 4. *If (M, ϕ) and (W, ψ) are a generalized time change one of* *the other. Then for a given* $\mu \in \mathcal{E}_\phi$ *define the probability* Γ_μ *by*

$$\int_M g d\Gamma_\mu = \frac{1}{\int_M \sigma(1,x)d\mu} \int_M \int_0^{\sigma(1,x)} g \circ \psi_s(P(x)) ds d\mu,$$

for all continuous map $g: W \to R$. *The probability* Γ_μ *is an ergodic* ψ-*invariant probability and the map* $\mu \to \Gamma_\mu$ *is a bijection from* $\mathcal{E}_\phi$ *onto* $\mathcal{E}_\psi$.

Proof. See [**8**].

4. Flow Entropy and Entropy of Time τ Maps

Lemma 5. *Let* ϕ *be a continuous flow which has an ergodic invariant probability* μ. *Then* $e_\mu(\phi) \leq \frac{1}{|\tau|} h_\mu(\phi_\tau)$, *for any* $\tau \in R\backslash\{0\}$. *In particular* $e_\mu(\phi) \leq h_\mu(\phi_1)$.

Proof. Let us consider three cases:

Case 1. Let us consider $\tau > 0$ and $t = n\tau$, where $n > 0$ is an integer.

For a given $\epsilon > 0$ take $\eta > 0$ such that $d(x, y) < \eta$ implies $d(\phi_s x, \phi_s y) < \epsilon$, if $0 \leq s \leq \tau$. For $x \in M$ we set

$$D(x, t, \epsilon, \phi) := \{y \in M;\ d(\phi_s x, \phi_s y) < \epsilon,\ 0 \leq s \leq t\}$$

and

$$\tilde{D}(x, n, \eta, \phi_\tau) := \{y \in M;\ d(\phi_{i\tau} x, \phi_{i\tau} y) < \eta,\ i = 0, 1, ..., n\}.$$

Then

$$\tilde{D}(x, n, \eta, \phi_\tau) \subset D(x, n\tau, \epsilon, \phi) \subset B(x, n\tau, \epsilon, \phi).$$

We denote by $\tilde{N}(\delta, n, \eta, \phi_\tau)$ the smallest number of open balls $\tilde{D}(x, n, \eta, \phi_\tau)$ needed to cover a set whose μ-probability is bigger than $1 - \delta$. We also recall that $N(\delta, t, \epsilon, \phi)$ denotes the smallest number of (t, ϵ, ϕ)-balls needed to cover a set whose μ-probability is bigger than $1 - \delta$. It follows that $N(\delta, n\tau, \epsilon, \phi) \leq \tilde{N}(\delta, n, \eta, \phi_\tau)$.

From [5], [**6**] it follows that

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \tilde{N}(\delta, n, \eta, \phi_\tau) = h_\mu(\phi_\tau)$$

therefore

$$e_\mu(\phi) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n\tau} \log N(\delta, n\tau, \epsilon, \phi) \leq \frac{1}{\tau} h_\mu(\phi_\tau).$$

Case 2. Let us consider $\tau > 0$ and $t > 0$.

Take $n_t > 0$ an integer such that $n_t\tau \le t < (n_t+1)\tau$. It is clear from the definition that $N(\delta,\, t,\, \epsilon,\, \phi) \le N(\delta,\, (n_t+1)\tau,\, \epsilon,\, \phi)$. Thus we get

$$\begin{aligned} e_\mu(\phi) &= \lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log N(\delta,\, t,\, \epsilon,\, \phi) \\ &\le \lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{n_t\tau} \log N(\delta,\, (n_t+1)\tau,\, \epsilon,\, \phi) \\ &= \lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{(n_t+1)\tau} \log N(\delta,\, (n_t+1)\tau,\, \epsilon,\, \phi) \\ &\le \frac{1}{\tau} h_\mu(\phi_\tau). \end{aligned}$$

Case 3. Let us consider $\tau < 0$ and $t > 0$.

Then taking $-\tau > 0$ and arguing like in Case 2 we get

$$e_\mu(\phi) \le -\frac{1}{\tau} h_\tau(\phi_{-\tau}) = -\frac{1}{\tau} h_\mu(\phi_\tau) = |\frac{1}{\tau}| h_\mu(\phi_\tau). \qquad \square$$

Theorem 5. *Let ϕ be a continuous flow on a compact metric space M. If ϕ has no fixed points and μ is an ergodic ϕ-invariant probability then $e_\mu(\phi) \ge |\frac{1}{\tau}| h_\mu(\phi_\tau)$, for any $\tau \in R \setminus \{0\}$.*

Proof. First we consider a partition $\xi = \{A_1, \dots, A_m, A_{m+1}\}$ of M such that

1. The sets $A_1, A_2, \dots, A_m$ are compact and pairwise disjoint.
2. $A_{m+1} = M \setminus (\bigcup_{i=1}^m A_i)$.

Then we define the sequence of partitions

$$\xi_\tau^n := \bigvee_{i=0}^{n-1} \phi_\tau^{-i} \xi$$

and recall that by definition

$$h_\mu(\phi_\tau, \xi) := -\lim_{n\to\infty} \sum_{A\in\xi_\tau^n} \mu(A) \log \mu(A).$$

The theorem is a consequence of the following claim.

Claim. For any $r > 0$ and any partition ξ satisfying Properties 1.-2. above it follows that

$$r + e_\mu(\phi) \ge \frac{1}{|\tau|} h_\mu(\phi_\tau, \xi).$$

In order to prove this claim we choose a positive integer L so that $\frac{1}{|\tau|L}\log 6 < r$ and consider three cases.

Case 1. Let us consider $\tau > 0$ and $t = nL\tau$, where $n > 0$ is an integer.

The element of $\xi^n_{L\tau}$ which contains x is denoted by $A_n(x)$. By Shannon-McMillan-Breiman theorem (see [4], [7], [9]) the limit

$$-\lim_{n\to\infty}\frac{1}{n}\log\mu(A_n(x))$$

exists for x in a set of full μ-probability. The sequence

$$x \to -\frac{1}{n}\log\mu(A_n(x))$$

converges in the L^1 norm to a L^1 function which we denote by $x \to h_\mu(\xi,\, \phi_{L\tau}, x)$. Since μ is by assumption ergodic and $h_\mu(\xi,\, \phi_{L\tau}, x) = h_\mu(\xi,\, \phi_{L\tau}, \phi_{L\tau}(x))$, it follows that, for x in a set of full μ-probability $h_\mu(\xi,\, \phi_{L\tau}, x) = h_\mu(\xi,\, \phi_{L\tau})$.

Take a small constant $b > 0$ and define

$$A_{nb}(\xi) := \{A \in \xi^n_{L\tau};\, \mu(A) < \exp(-n(h_\mu(\xi,\, \phi_{L\tau}) - b))\}$$

and

$$\mathcal{A}_{nb}(\xi) := \bigcup_{A\in A_{nb}(\xi)} A.$$

It follows that $\mu(\mathcal{A}_{nb}(\xi)) > 2\delta$ for some $\delta > 0$ and all n big enough.

Set $\eta_0 := \min\{d(x,y); x \in A_i, y \in A_j \text{ and } 1 \le i \ne j \le m\}$. Given $\eta \in (0,\eta_0)$ we choose $\theta > 0$ so that $d(\phi_s(z),\, z) < \eta/3$ for all $z \in M$ and $|s| \le \theta$. We also choose $\epsilon \in (0,\eta/3)$ corresponding to $\epsilon_1 = \theta/(4L\tau)$ in Lemma 3. Then we set $N := N(\delta,\, t,\, \epsilon,\, \phi)$ and consider $(t,\, \epsilon,\, \phi)-$balls $B(x_1,\, t,\, \epsilon,\, \phi)$, ..., $B(x_N,\, t,\, \epsilon,\, \phi)$ whose union covers a set of μ-probability bigger than $1-\delta$. Observe that

$$\mu(\mathcal{A}_{nb}(\xi) \cap \bigcup_{j=1}^{N} B(x_j,\, t,\, \epsilon,\, \phi)) > \delta.$$

Let us prove now that for each $j = 1,\dots,N$ at most 6^n elements from $A_{nb}(\xi)$ have non-empty intersection with $B(x_j,\, t,\, \epsilon,\, \phi)$. Indeed, if $x \in A\bigcap B(x_j,\, t,\, \epsilon,\, \phi)$ there exists $\alpha \in Rep[0,\, t]$ such that

$$d(\phi_{\alpha(s)}x_j,\, \phi_s x) < \epsilon, \quad 0 \le s \le t.$$

Setting $u := s-s_1$ and $\gamma(u) := \alpha(s)-\alpha(s_1)$ we get $\gamma \in Rep[-s_1,\, t-s_1]$ such that $d(\phi_{\gamma(u)}\phi_{\alpha(s_1)}x_j,\, \phi_u\phi_{s_1}x) = d(\phi_{\alpha(s)}x_j,\, \phi_s x) < \epsilon$, for $-s_1 \leq u \leq t-s_1$. So, for $u = s_2 - s_1$ with $|s_2 - s_1| \leq L\tau$ it follows from Lemma 3 that $|(\alpha(s_1) - s_1) - (\alpha(s_2) - s_2)| \leq \theta/4$.

Then we denote by $[z]$ the biggest integer smaller or equal to z and consider the following sequence of integer numbers

$$S_\alpha := \{[\frac{\alpha(kL\tau) - kL\tau}{\frac{\theta}{4}}]\},\ k = 0,\, 1,\, \dots,\, n-1.$$

If for another element $\tilde{A} \in \xi^n_{L\tau}$ there exists $y \in \tilde{A} \bigcap B(x_j,\, t,\, \epsilon,\, \phi)$, then we can take $\beta \in Rep[0,t]$ such that $d(\phi_{\beta(s)}x_j,\, \phi_s y) < \epsilon,\quad 0 \leq s \leq t$.

If the sequences S_α and S_β are the same we get

$$\begin{aligned}
|\alpha(s) - \beta(s)| \leq\ & |(\alpha(s) - s) - (\alpha([\frac{s}{L\tau}]L\tau) - [\frac{s}{L\tau}]L\tau)| \\
& + |(\alpha([\frac{s}{L\tau}]L\tau) - [\frac{s}{L\tau}]L\tau) - (\beta([\frac{s}{L\tau}]L\tau) - [\frac{s}{L\tau}]L\tau)| \\
& + |(\beta(s) - s) - (\beta([\frac{s}{L\tau}]L\tau) - [\frac{s}{L\tau}]L\tau)| \\
& \leq \frac{\theta}{4} + \frac{\theta}{4}|\frac{\alpha([\frac{s}{L\tau}]L\tau) - [\frac{s}{L\tau}]L\tau}{\frac{\theta}{4}} - \frac{\beta([\frac{s}{L\tau}]L\tau) - [\frac{s}{L\tau}]L\tau}{\frac{\theta}{4}}| + \frac{\theta}{4} \\
& \leq \theta,
\end{aligned}$$

for all $s \in [0,t]$. From the choice of θ it follows that $d(\phi_{\alpha(s)}x_j,\, \phi_{\beta(s)}x_j) < \eta/3$ for all $s \in [0,t]$. Therefore

$$\begin{aligned}
d(\phi_s x,\, \phi_s y) &\leq d(\phi_s x,\, \phi_{\alpha(s)}x_j) + d(\phi_{\alpha(s)}x_j,\, \phi_{\beta(s)}x_j) + d(\phi_{\beta(s)}x_j,\, \phi_s y) \\
&< \epsilon + \frac{\eta}{3} + \epsilon < \eta
\end{aligned}$$

for all $0 \leq s \leq t$. In particular $d(\phi^i_{L\tau}x,\, \phi^i_{L\tau}y) \leq \eta,\quad i = 0,\, 1,\, \dots,\, n-1$.

Recall that for an element A in $A_{nb}(\xi)$ there exist $i_0, i_1, \dots, i_{n-1} \in \{1, \dots, m+1\}$ such that

$$A = A_{i_0} \bigcap \phi^{-1}_{L\tau}(A_{i_2}) \bigcap \cdots \bigcap \phi^{-(n-1)}_{L\tau}(A_{i_{n-1}}).$$

Then for a given sequence S_α there exist at most 2^n choices for A such that $A \cap B(x_j,\, t,\, \epsilon,\, \phi) \neq \varnothing$.

Now observe that the first term of a sequence S_α is zero and two consecutive terms of it differ at most by 1. So there exist at most 3^{n-1}

such sequences. Then we conclude that for each $j = 1, \dots, N$ at most 6^n elements from $A_{nb}(\xi)$ has non-empty intersection with $B(x_j, t, \epsilon, \phi)$. It follows that at most $N6^n$ elements from $A_{nb}(\xi)$ has non-empty intersection with $\bigcup_{j=1}^N B(x_j, t, \epsilon, \phi)$.

Recall that

$$\mu(A_{nb}(\xi) \cap \bigcup_{j=1}^{N} B(x_j, t, \epsilon, \phi)) > \delta$$

and for each $A_{nb}(x)$ in $A_{nb}(\xi)$

$$\mu(A_{nb}(x) \cap \bigcup_{1}^{N} B(x_j, t, \epsilon, \phi)) < \exp(-n(h_\mu(\phi_{L\tau}, \xi) - b)).$$

It follows that at least $\delta \exp(n(h_\mu(\phi_{L\tau}, \xi) - b))$ elements from $A_{nb}(\xi)$ has non-empty intersection with $\bigcup_{j=1}^N B(x_j, t, \epsilon, \phi)$. Therefore

$$6^n N(\delta, t, \epsilon, \phi) \geq \delta \exp(n(h_\mu(\phi_{L\tau}, \xi) - b)).$$

It follows that

$$\frac{1}{\tau} h_\mu(\phi_\tau, \xi) \leq e_\mu(\phi) + r$$

as we claimed.

case 2. Let us consider $\tau > 0$ and $t > 0$.

Take $n_t \in Z^+$ such that $n_t L\tau \leq t < (n_t + 1)L\tau$. It is clear from the definition that $N(\delta, t, \epsilon, \phi) \geq N(\delta, n_t L\tau, \epsilon, \phi)$. Thus we get

$$\begin{aligned} e_\mu(\phi) + r &= \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log N(\delta, t, \epsilon, \phi) + r \\ &\geq \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{(n_t + 1)L\tau} \log N(\delta, n_t L\tau, \epsilon, \phi) + r \\ &= \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{n_t L\tau} \log N(\delta, n_t L\tau, \epsilon, \phi) + r \\ &\geq \frac{1}{\tau} h_\mu(\phi_\tau, \xi). \end{aligned}$$

Case 3. Let us consider $\tau < 0$ and $t > 0$.

Then taking $-\tau > 0$ and arguing like in Case 2 we get

$$e_\mu(\phi) \geq -\frac{1}{\tau} h_\tau(\phi_{-\tau}) = -\frac{1}{\tau} h_\mu(\phi_\tau) = |\frac{1}{\tau}| h_\mu(\phi_\tau). \qquad \square$$

Corollary 1. *Let ϕ be a flow on a compact metric space which has no fixed points. Then for any ergodic ϕ-invariant probability μ we have that*

$$h_\mu(\phi_1) = e_\mu(\phi) = \lim_{\epsilon\to 0}\lim_{t\to\infty}\frac{1}{t}\log N(\delta,\, t,\, \epsilon,\, \phi).$$

Proof. We can replace lim sup by lim inf in the proof of Theorem 5 and get

$$\lim_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t}\log N(\delta,\, t,\, \epsilon,\, \phi) \geq h_\mu(\phi_1).$$

This together with Lemma 5 imply that $h_\mu(\phi_1) = e_\mu(\phi)$. □

5. Proofs of Theorems 1-3

Proof of Theorem 1. Let $(M,\, \phi,\, \mu)$ and $(W,\, \psi,\, \nu)$ be measure-theoretically equivalent flows where μ, ν are ergodic. Let P, σ be as in Definition 2

Lemma 1 guarantees the existence of a constant $M_2 > 0$ such that $0 \leq \sigma_x(t) \leq M_2 t$ for all $x \in M$ and $t \geq 1$, see [**10**].

For a given $\epsilon > 0$ choose $\eta > 0$ such that $d(P^{-1}(y_1),\, P^{-1}(y_2)) < \epsilon$ for all $y_1,\, y_2 \in W$ with $d(y_1, y_2) < \eta$. Let us fix $\delta > 0$, $N := N(\delta,\, t,\, \eta,\, \psi)$ and choose $(t,\, \eta,\, \psi)-$balls $B(y_1,\, t,\, \eta,\, \psi), ..., B(y_N,\, t,\, \eta,\, \psi)$ whose union covers a set of ν-probability bigger than $1 - \delta$.

For $y \in B(y_j,\, t,\, \eta,\, \psi)$ there is $\alpha \in Rep[0, t]$ such that

$$d(\psi_{\alpha(s)}(y_j),\, \psi_s(y)) < \eta,\ \ 0 \leq s \leq t.$$

Taking $\lambda_y = \sigma_x^{-1}$ where $y = P(x)$ it follows that

$$\begin{aligned} & d(\phi_{\lambda_{y_j}(\alpha(s))} \circ P^{-1}(y_j),\, \phi_{\lambda_y(s)} \circ P^{-1}(y)) = \\ & = d(P^{-1} \circ \psi_{\alpha(s)}(y_j),\, P^{-1} \circ \psi_s(y)) < \epsilon,\ \ 0 \leq s \leq t. \end{aligned}$$

Setting $u := \lambda_y(s)$, $\beta(u) := \lambda_{y_j} \circ \alpha \circ \lambda_y^{-1}(u)$ and recalling that $\lambda_y(t) = \sigma_x^{-1}(t)$ it follows that $\lambda_y(t) \geq \frac{1}{M_2} t$ for all $t \geq 1$. Then

$$d(\phi_{\beta(u)} \circ P^{-1}(y_j),\, \phi_u \circ P^{-1}(y)) < \epsilon,\ \ 0 \leq u \leq \frac{t}{M_2}.$$

Therefore

$$P^{-1}(B(y_j,\, t,\, \eta,\, \psi)) \subset B(P^{-1}(y_j),\, \frac{t}{M_2},\, \epsilon,\, \phi),\ \ j = 1,\, 2,\, ...,\, N.$$

Since $P_*\mu = \nu$ we get

$$\mu(\bigcup_{j=1}^{N} B(P^{-1}(y_j), \frac{t}{M_2}, \epsilon, \phi)) \geq \nu(\bigcup_{i=1}^{N} B(y_i, t, \eta, \psi)) \geq 1 - \delta$$

and

$$N(\delta, \frac{t}{M_2}, \epsilon, \phi) \leq N(\delta, t, \eta, \psi).$$

For $B_2 := M_2$ it follows that $e_\mu(\phi) \leq B_2 e_\nu(\psi)$. By symmetry there exists another constant B_1 such that $e_\mu(\phi) \geq B_1 e_\nu(\psi)$ and theorem follows immediately. □

Proof of Theorem 2. This theorem follows immediately from Lemma 5 and Theorem 5. □

Proof of Theorem 3.

Part 1. By Lemma 1 there exists a constant $M_2 > 0$ such that $\sigma(x, t) \leq M_2 t$ for all $x \in M$ and $t \geq 1$. For a given $\epsilon > 0$ choose $\eta > 0$ so that $d(P^{-1}(y_1), P^{-1}(y_2)) < \epsilon$ for all $y_1, y_2 \in W$ with $d(y_1, y_2) < \eta$.

Given $\mu \in \mathcal{E}_\phi$ set

$$\beta := \frac{\sup\{\sigma(x,1); x \in M\}}{\int_M \sigma(x,1) d\mu},$$

clearly $\beta \geq 1$. If $\Gamma_\mu \in \mathcal{E}_\phi$ is the probability given by Lemma 4 and B is a Borel set we have that

$$\begin{aligned}\Gamma_\mu(P(B)) &= \frac{1}{\int_M \sigma(x,1) d\mu} \int_M \left(\int_0^{\sigma(x,1)} \chi_{P(B)} \circ \psi_s(P(x)) ds \right) d\mu \\ &= \frac{\int_B \sigma(x,1) d\mu}{\int_M \sigma(x,1) d\mu},\end{aligned}$$

here $\chi_{P(B)}$ denotes the characteristic function of the set $P(B)$. Then $\Gamma_\mu(P(B)) \leq \beta\mu(B)$.

Let us fix $\delta > 0$, $N := N(\delta, t, \eta, \psi)$ and choose (t, η, ψ)-balls

$$B(y_1, t, \eta, \psi), ..., B(y_N, t, \eta, \psi)$$

whose union covers a subset of W with Γ_μ-probability bigger than $1 - \delta$. Since

$$P^{-1}(B(y_i, t, \eta, \psi)) \subset B(P^{-i}(y_i), \frac{t}{M_2}, \epsilon, \phi), \;\; i = 1, ..., N$$

it follows that

$$\mu(\bigcup_{i=1}^{n} B(P^{-1}(y_i), \frac{t}{M_2}, \epsilon, \psi)) \geq \mu(P^{-1}(\bigcup_{i=1}^{n} B(y_i, t, \eta, \psi))) \geq (1-\delta)/\beta =: 1-\delta'.$$

So

$$N(\delta', \frac{t}{M_2}, \epsilon, \phi) \leq N(\delta, t, \eta, \psi).$$

Since $e_\mu(\phi)$ is not dependent on the choice of δ we get $e_\mu(\phi) \leq M_2 e_{\Gamma_\mu}(\psi)$. Taking $C_2 := M_2$ we have that $e(\phi) \leq C_2 e(\psi)$. By symmetry we also get $C_1 e(\psi) \leq e(\phi)$, for some constant $C_1 > 0$ and Part 1 of the theorem follows.

Part 2. Let us remember that the set of all ergodic ϕ_t-invariant and ϕ-invariant probabilities are denoted respectively by $\mathcal{E}_{\phi_t}$ and $\mathcal{E}_\phi$. By $\mathcal{M}_{\phi_t}$ we denote the set of all ϕ_t-invariant probabilities.

It is clear that $\mathcal{E}_\phi \subset \mathcal{M}_{\phi_t}$. From Theorem 2 we get

$$e(\phi) = \sup_{\mu\in\mathcal{E}_\phi} e_\mu(\phi) \leq \sup_{\mu\in\mathcal{E}_\phi} h_\mu(\phi_1) \leq \sup_{\nu\in\mathcal{M}_{\phi_1}} h_\nu(\phi_1) = h(\phi_1).$$

If the flow ϕ has no fixed points it follows from Theorem 5 that

$$\sup_{\mu\in\mathcal{E}_\phi} e_\mu(\phi) \geq \sup_{\mu\in\mathcal{E}_\phi} h_\mu(\phi_1).$$

In [**8**] it is proved that

$$\sup_{\mu\in\mathcal{E}_\phi} h_\mu(\phi_1) \geq \sup_{\nu\in\mathcal{E}_{\phi_1}} h_\nu(\phi_1) = h(\phi_1).$$

This part of the theorem follows immediately. □

6. Expansive Flows and Symbolic Dynamics

Let us start this section recalling some facts from [**3**]. Given a finite family $\mathcal{F} = \{S_1, \ldots, S_k\}$ we set $\Sigma_\mathcal{F} := \prod_Z \mathcal{F}$. The elements of $\Sigma_\mathcal{F}$ are bi-infinite sequences which we denote by $S = \{S_i\}_{i=-\infty}^{\infty}$. The metric d in $\Sigma_\mathcal{F}$ is defined as follows

$$d(S^1, S^2) := \sum_{i=-\infty}^{\infty} \frac{\delta(S_i^1, S_i^2)}{2^{-|i|}},$$

where $\delta(S_i^1, S_i^2) = 0$ if $S_i^1 = S_i^2$ and $\delta(S_i^1, S_i^2) = 1$ if $S_i^1 \neq S_i^2$.

The map $\sigma: \Sigma_{\mathcal{F}} \to \Sigma_{\mathcal{F}}$ is the shift defined by $\sigma(S) = \tilde{S}$ where $\tilde{S}_i = S_{i+1}$. Defined in this way σ is an expansive homeomorphism of $\Sigma_{\mathcal{F}}$.

Definition 3. *A flow ϕ is called* expansive *if for any $\varepsilon > 0$ there exists $\vartheta > 0$ so that, if $d(\phi_s(x), \phi_{\alpha(s)}(y)) < \vartheta$ for some $x, y \in M$, $\alpha \in Rep(R)$ or $\alpha \equiv 0$ and any $s \in R$ it follows that $y = \phi_t(x)$ with $|t| < \varepsilon$.*

The definition of expansive flow is independent on the choice of the metric. Among expansive flows there are Anosov flows, Smale Axiom A flows and suspensions of expansive homeomorphisms. Expansive flows have finitely many fixed points and each one of them is an isolated point of M. This reduces the study of expansive flows to those without fixed points.

Throughout this section we assume that $\phi: R \times M \to M$ is an expansive flow without fixed points.

Definition 4. *Given a flow ϕ and $\zeta > 0$ a* local cross section at time ζ *is a closed set S contained in M such that $S \cap \phi_{[-\zeta,\zeta]}(x) = \{x\}$ for all $x \in S$.*

If S is a local cross section of ϕ at time ζ we have that ϕ maps $S \times [-\zeta, \zeta]$ homeomorphicaly onto $\phi_{[-\zeta,\zeta]}(S)$. Defining

$$S^* := S \bigcap \mathrm{Int}\,(\phi_{[-\zeta,\zeta]}(S))$$

(for any $s > 0$) it follows that $\phi_{(-s,s)}(S^*)$ is an open set and $\phi_{[-s,s]}(S \backslash S^*)$ is a closed set with empty interior.

It is proved in [**3**] that there exist $\varepsilon > 0$, $\vartheta \in (0, \varepsilon)$ and a family $\mathcal{F} = \mathcal{F}(\varepsilon, \vartheta) = \{S_1, \dots, S_k\}$ such that

1. S_1, S_2, ..., S_k are pairwise disjoint local cross sections at time ε.
2. $\vartheta > 0$ is the constant corresponding to $\varepsilon > 0$ given by Definition 3 and diam $\mathcal{F} := \max\{\text{diam } S_i;\ \ i = 1, \dots, k\}$ is smaller than ϑ.
3. $M = \phi_{[0,\vartheta]} \bigcup_{i=1}^k S_i = \phi_{[-\vartheta,0]} \bigcup_{i=1}^k S_i$.
4. $\phi_{(0,b)}(x) \cap \bigcup_{i=1}^k S_i = \varnothing$ and $\phi_{(-b,0)}(x) \cap \bigcup_{i=1}^k S_i = \varnothing$ for all $x \in \bigcup_{i=1}^k S_i$ and some $b \in (0, \vartheta)$.

Define

$$W := M \setminus \bigcup_{n\in Z} \phi_{n\vartheta}\phi_{[-\vartheta,\vartheta]}(\bigcup_{i=1}^{k} S_i \setminus S_i^*) \quad \text{and} \quad V := W \cap \bigcup_{i=1}^{k} S_i.$$

For $x \in V$ we consider the doubly infinite sequence

$$\cdots < t_{-2}(x) < t_{-1}(x) < t_0(x) = 0 < t_1(x) < t_2(x) < \dots$$

of all t such that $\phi_t(x) \in \bigcup_{i=1}^{k} S_i$. Let $Q_i(x)$ denote the element of $\mathcal{F}$ that $\phi_{t_i(x)}(x)$ belongs to and define $Q(x) = \{Q_i(x)\}_{i=-\infty}^{+\infty}$. Then we get a σ-invariant closed set $\Lambda_{\mathcal{F}} = \overline{Q(V)} \subset \Sigma_{\mathcal{F}}$.

For $T = \{T_i\}_{-\infty}^{+\infty}$ in $\Lambda_{\mathcal{F}}$ we take $x \in \bigcup_{i=1}^{k} S_i$ and a doubly infinite sequence $\{t_i(x)\}_{-\infty}^{+\infty}$ with $t_0(x) = 0$, $t_{i+1}(x) - t_i(x) \in [b, \vartheta]$ so that $\phi_{t_i(x)}(x) \in T_i$. Then, defining $f(T) := t_1(x)$ we get a continuous well defined function $f: \Lambda_{\mathcal{F}} \to R$ such that $f(T) \geq b$. A symbolic suspension flow for ϕ under f, namely $(\Lambda_{\mathcal{F}}^f, \varphi)$ or just φ, can be defined as follows

$$\Lambda_{\mathcal{F}}^f := \{(T,s); T \in \Lambda_{\mathcal{F}},\ 0 \leq s \leq f(T)\}/(T, f(T)) \sim (\sigma(T), 0)$$

and

$$\varphi((T,s),t) = \varphi_t(T,s) = (T, t+s), \quad 0 \leq t+s < f(T).$$

There exists a continuous surjection $\rho: \Lambda_{\mathcal{F}}^f \to M$ so that $\rho \circ \varphi_t = \phi_t \circ \rho$ for all t. Moreover, there exists a Baire set $\tilde{W} = \rho^{-1}(W)$ contained in $\Lambda_{\mathcal{F}}^f$ which is mapped by ρ homeomorphicaly onto the Baire set W. See [**3**] for more details.

In [**2**] it is proved that $h(\phi_1) = h(\varphi_1)$ in the case that ϕ is an Axiom A flow. In [**3**] the following problem is posed.

Problem. In the case that ϕ is an expansive flow is it possible to get $h(\phi_1) = h(\varphi_1)$ by choosing the family $\mathcal{F}$ of local cross sections carefully?

The following theorem assures a positive answer to this problem.

Theorem 4. *Let ϕ be an expansive flow without fixed points and φ be a symbolic suspension flow for ϕ, then $h(\phi_1) = h(\varphi_1)$.*

To prove this theorem we need the following lemma.

Lemma 6. *Given $\epsilon > 0$ there exists $\eta > 0$ so that for all $y_1, y_2 \in W$ with $d(y_1, y_2) < \eta$ we have that $d(\rho^{-1}(y_1), \rho^{-1}(y_2)) < \frac{\epsilon}{4}$.*

Proof. We recall that $\rho^{-1}_{|W}$ is a homeomorphism between the Baire sets W and $\tilde{W}$ and consider the following claim.

Claim. For any a in M and any $A \in \rho^{-1}(a)$, there exists $\eta(a) > 0$ such that $\rho^{-1}(B(a, \eta(a))) \subset B(A, \frac{\epsilon}{4})$.

To prove this claim we consider two cases.

Case 1. In this case we assume that $a \in W$. By the continuity of ρ^{-1} in W there exists $\eta(a) > 0$ such that

$$\rho^{-1}(B(a, \eta(a)) \cap W) \subset B(A, \frac{\epsilon}{8}).$$

This implies that $\rho^{-1}B(a, \eta(a)) \subset B(A, \epsilon/4)$. In fact, otherwise the open set $\rho^{-1}(B(a, \eta(a))) \setminus \overline{B}(A, \frac{\epsilon}{8})$ would contain some $T \in \Lambda^f_{\mathcal{F}}$ and its neighborhood $B(T, l)$ for some small $l > 0$. Since $\tilde{W}$ is dense in $\Lambda^f_{\mathcal{F}}$ one can find a point P in $B(T, l) \cap \tilde{W}$ what contradicts $P = \rho^{-1}(\rho(P)) \in \rho^{-1}(B(a, \eta(a)) \cap W) \subset B(A, \frac{\epsilon}{8})$. Therefore the claim is true

Case 2. In this case we assume that $a \in M \setminus W$. Then we take and fix $A \in \rho^{-1}(a)$ and notice that $\rho(B(A, \frac{\epsilon}{8}))$ contains not only $a = \rho(A)$ but a neighborhood of a in M. In fact, otherwise for each positive integer n one could pick out $y_n \in (B(a, \frac{1}{n}) \cap W) \setminus \rho(B(A, \frac{\epsilon}{8}))$. The sequence y_n converges to a as n tends to infinity. Take a sequence $\{A_n\}_{n=1}^{\infty}$ in $\tilde{W}$ such that A_n converges to A. Then $d(y_n, \rho(A_n))$ converges to zero as n tends to infinity. Since $\rho^{-1}_{|W}$ is a homeomorphism between the Baire sets W and $\tilde{W}$ we get that $d(\rho^{-1}(y_n), A_n)$ converges to zero. So $\rho^{-1}(y_n) \in B(A, \frac{\epsilon}{8})$ for n large enough which contradicts the choice of y_n. This implies that $\rho(B(A, \frac{\epsilon}{8}))$ contains $B(a, \eta(a))$ for some $\eta(a) > 0$. Clearly $\rho^{-1}(B(a, \eta(a)) \cap W) \subset B(A, \frac{\epsilon}{8})$. Again the claim is true.

In this way we get an open cover $\{B(a, \eta(a)); a \in M\}$ of M. If $\eta > 0$ is the Lebesgue number of this cover it follows that $\rho^{-1}(B(y, \eta)) \subset B(\rho^{-1}y, \frac{\epsilon}{4})$, for any $y \in M \cap W$ and the lemma follows. □

Proof of Theorem 4. The notation employed in this proof is the same introduced above. Take ν an ergodic probability for the flow φ on $\Lambda^f_{\mathcal{F}}$ and set $\mu := \rho_* \nu$. Then μ is an ergodic probability for the flow ϕ on M.

Given $x \in M$, and t, $\eta \in R$ we set

$$D(x,\, t,\, \eta,\, \phi) := \{y \in M;\ d(\phi_s(x),\, \phi_s(y)) < \eta,\, 0 \le s \le t\}.$$

Given $\delta > 0$ denote by $\tilde{N}(\delta,\, t,\, \eta,\, \phi)$ the smallest number of these open sets needed to cover a subset of μ−probability bigger than $1 - \delta$. Let us prove that $\tilde{N}(\delta, t, \eta, \phi) \ge N(\delta, t, \varepsilon, \varphi)$, see Definition 1 in order to remember the meaning of the notation $N(\delta, t, \varepsilon, \varphi)$. Indeed, if $N := \tilde{N}(\delta, t, \eta, \phi)$ we choose $D(x_1, t, \eta, \phi), \ldots, D(x_N, t, \eta, \phi)$ whose union cover a set of μ-probability bigger than $1 - \delta$.

First we need to replace the points x_i possibly not in W to points z_i in W. Then let ε, η be as in Lemma 6 and take $\xi > 0$ very small so that, if $d(y, y') < \eta + \xi$ then $d(\rho^{-1}(y), \rho^{-1}(y')) < \varepsilon/2$, for any $y, y' \in W$. Take $\omega > 0$ small enough so that, if $d(y, y') < \omega$ then $d(\phi_s(y), \phi_s(y')) < \xi$, for any $y, y' \in M$ and $0 \le s \le t$.

Now we choose $z_i \in B(x_i, \omega) \cap D(x_i, t, \eta, \phi) \cap W$, where $B(x_i, \omega)$ denotes the open ball of radius ω centered at x_i. For $y \in D(x_i, t, \eta, \phi)$ one sees that

$$d(\phi_s(z_i), \phi_s(y)) \le d(\phi_s(z_i), \phi_s(x_i)) + d(\phi_s(x_i), \phi_s(y)) < \eta + \xi, \quad 0 \le s \le t.$$

This gives $D(x_i, t, \eta, \phi) \subset D(z_i, t, \eta+\xi, \phi)$, $\quad i = 1, \ldots, N$. Then the open sets $D(z_1, t, \eta + \xi, \phi), \ldots, D(z_N, t, \eta + \xi, \phi)$ cover a set of μ-probability bigger than $1 - \delta$.

From Lemma 6 it follows that

$$\rho^{-1}(D(z_i, t, \eta + \xi, \phi) \cap W) \subset B(\rho^{-1} z_i, t, \varepsilon/2, \varphi),$$

and, since $\rho^{-1}(W)$ is dense in $\Lambda^f_{\mathcal{F}}$ we see that

$$\rho^{-1} D(z_i, t, \eta + \xi, \phi) \subset B(\rho^{-1} z_i, t, \varepsilon, \varphi), \quad i = 1, 2, \ldots, N.$$

Then

$$\nu(\bigcup_{i=1}^{N} B(\rho^{-1} z_i, t, \varepsilon, \varphi) \ge \nu(\rho^{-1}(\bigcup_{i=1}^{N} D(x_i, t, \eta, \phi)$$
$$= \mu(\bigcup_{i=1}^{N} D(x_i, t, \eta, \phi) > 1 - \delta.$$

It follows that $\tilde{N}(\delta, t, \eta, \phi) \ge \tilde{N}(\delta, t, \varepsilon, \varphi)$.

Now we choose $\alpha > 0$ small enough so that $d(y, y') < \alpha$ implies $d(\phi_s(y), \phi_s(y')) < \eta$, $0 \le s \le 1$. If $t = (k-1) + q$, $k \in Z$ and $0 \le q < 1$ we set

$$D(y, k, \alpha, \phi_1) := \{y' \in M;\ d(\phi_i(y), \phi_i(y')) < \alpha, \quad i = 0, 1, ..., k-1\}.$$

If $R(\delta, k, \alpha, \phi_1)$ denotes the smallest number of these open sets needed to cover a subset of μ−probability bigger than $1-\delta$ then

$$R(\delta, k, \alpha, \phi_1) \ge \tilde{N}(\delta, t, \eta, \phi) \ge N(\delta, t, \epsilon, \varphi).$$

From this and the definition of $h_\mu(\phi_1)$ and $e_\mu(\phi)$ it follows that $h_\mu(\phi_1) \ge e_\nu(\varphi)$.

One can easily check that $\rho_* \mathcal{E}_\varphi = \mathcal{E}_\phi$, so by Theorem 2 together with the fact that neither ϕ nor φ has fixed points we get

$$\begin{aligned} e(\phi) &= \sup\{e_\mu(\phi);\ \ \mu \in \mathcal{E}_\phi\} \\ &= \sup\{h_\mu(\phi_1); \mu \in \mathcal{E}_\phi\} \\ &\ge \sup\{e_\nu(\varphi);\ \ \nu \in \mathcal{E}_\varphi\} = e(\varphi). \end{aligned}$$

Then by Theorem 3 it follows that $h(\phi_1) = e(\phi) \ge e(\varphi) = h(\varphi_1)$. From $\rho \circ \varphi_1 = \phi_1 \circ \rho$ one can easily show that $h(\phi_1) \le h(\varphi_1)$ and then we get $h(\phi_1) = h(\varphi_1)$. □

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